Holomorphic Cliffordian functions as a natural extension of monogenic and hypermonogenic functions

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This is an expository paper which aim is to defend the notion of holomorphic Cliffordian functions [11], [12]. The way to argue for is to exhibit non-trivial applications. Some of them were known earlier [13], [14]. The most recent interesting development was the contribution [4] in which clearly one see how holomorphic Cliffordian functions are able to solve a problem which was unsatisfactory solved before.

The organization of this paper is almost the same as in [20]. In section 1., we recall the fundamental definitions and properties of Clifford algebras, especially those of anti-euclidean type. The sections 2., 3. and 4. are devoted to a brief overview of the different theories of "hypercomplex" variables, namely the classical theory of monogenic functions, the holomorphic Cliffordian functions, and finally the hypermonogenic functions, respectively. A careful analysis of the connections between those three classes of functions argues for the holomorphic Cliffordian ones. This is a set which is endowed with many function theoretical tools that are also offered for complex holomorphic functions. Basically, they were introduced in order to contain the functions $x \mapsto x^n (n \in \mathbb{N}, x$ a paravector) and to be stable under any directional derivation. Consequently, they form a class of functions containing the two others.

They are two new sections. In section 5., we deal with the problem how to construct analogues of the Weierstrass $\zeta$ and $\wp$ functions, as well as the Jacobi $cn$, $dn$ and $sn$. We end with the section 6., deeply inspired of [4], where we are illustrating a new class of Clifford valued automorphic forms on arithmetic subgroups of the Ahlfors-Vahlen group.

1. CLIFFORD ALGEBRAS

Denote by $\mathbb{R}^{p+q}$ a real vector space of dimension $d = p + q$ provided with a non-degenerate quadratic form $Q$ of signature $(p, q)$. 

Main definition. The Clifford algebra, we will denote by $R_{p,q}$, of the quadratic form $Q$ on the vector space $R^{p+q}$ is an associative algebra over $R$, generated by $R^{p+q}$, with unit 1, if it contains $R$ and $R^{p+q}$ as distinct subspaces and

(1) $\forall v \in R^{p+q}, v^2 = Q(v),
(2)$ the algebra is not generated by any proper subspace of $R^{p+q}$.

Actually, if we consider the Clifford algebra $R_{p,q}$ as a vector space, it has the splitting: $R_{p,q} = R_{p,q}^0 \oplus R_{p,q}^1 \oplus ... \oplus R_{p,q}^k \oplus ... \oplus R_{p,q}^d$, where $R_{p,q}^0 = R$ are the scalars, $R_{p,q}^k = R^{p+q}$ is the vector space, $R_{p,q}^2$ is the vector space of the so-called bivectors corresponding to the planes in $R^{p+q}$, and so on. Finally, $R_{p,q}^d$ contains what we call the pseudoscalars. Moreover,

$$dim_R R_{p,q}^k = C_d^k, \quad dim_R R_{p,q} = 2^d.$$

Now, set $e_0 = 1$ as basis of $R_{p,q}^0 = R$ and suppose $\{e_1, e_2, ..., e_d\}$ be an orthonormal basis for $R_{p,q}^1 = R^{p+q}$. Thus, the corresponding vector spaces of the splitting will be provided with respective basis $\{e_0 = 1\}, \{e_1, e_2, ..., e_d\}$, $\{e_{ij} = e_i e_j, 1 \leq i < j \leq d\}, ..., \{e_{i_1...i_k} = e_{i_1} e_{i_2} ... e_{i_k}, 1 \leq i_1 < i_2 < ... < i_k \leq d\}, ..., \{e_{d} = e_1 e_2 ... e_d\}$, and the algebra will obey to the laws:

$$e_i^2 = 1, i = 1, ..., p, e_i^2 = -1, i = p+1, ..., d = p+q, e_i e_j = -e_j e_i, i \neq j.$$

This allows us to write down any Clifford number $a \in R_{p,q}$ as a sum of its scalar part $<a>_0$, its vector part $<a>_1 \in R_{p,q}^1$, its bivector part $<a>_2 \in R_{p,q}^2$, up to its pseudoscalar part $<a>_d \in R_{p,q}^d$, namely

$$a = <a>_0 + <a>_1 + ... + <a>_d,$$

where $<a>_k = \sum |J|=k a_J e_J$, with $J = (j_1, ..., j_k)$ is a strictly increasing multiindice of length $k$ and $e_J = e_{j_1} e_{j_2} ... e_{j_k}$, while $a_J \in R$.

Some examples: The Clifford algebra $R_{0,1}$ can be identified with the complex numbers $C$. The algebra $R_{0,2}$ is nothing else than the set of quaternions $H$ if we identify $e_1 = i, e_2 = j, e_{12} = k$ using the traditional notations. Physicists are working very often with the algebras $R_{1,3}$ or $R_{3,1}$. It suffices to note the nature of the corresponding signatures $(+, -, -, -)$ and $(+, +, +, -)$, respectively. However, $R_{1,3}$ and $R_{3,1}$ are not isomorphic as algebras.

Recall that the main involution $'$, the reversion anti-automorphism $\sim$ and the conjugation anti-automorphism $-$ act on $a \in R_{p,q}$ as follows:

$$a' = \sum_{k=0}^{d} (-1)^{k} <a>_k, \quad a' = \sum_{k=0}^{d} \frac{(-1)^{k(k-1)}}{2} <a>_k,$$
and
\[ \bar{a} = \sum_{k=0}^{d} (-1)^{\frac{k(k+1)}{2}} < a >_k \]

We also need the following automorphism \( * : R_{0,n} \to R_{0,n} \) defined by the relations: \( e_n^* = -e_n, e_i^* = e_i \) for \( i = 0, 1, ..., n-1 \) and \( (ab)^* = a^*b^* \) for \( a, b \in R_{0,n} \). These operations are not of algebraic type, they are of geometric type.

**Remarks.** For \( C = R_{0,1} \), the usual complex conjugation is the main involution, as well as the conjugation. The reversion is useless.

For \( H = R_{0,2} \), with the classical notations, we have: \( a = \alpha + \beta i + \gamma j + \delta k, a' = \alpha - \beta i - \gamma j + \delta k, a' = \alpha + \beta i + \gamma j - \delta k, \bar{a} = \alpha - \beta i - \gamma j - \delta k \).

Henceforth, we will consider Clifford algebras of antiEuclidean type, namely \( R_{0,d} \), ([5], [2]). Note the first three, for \( d = 0, 1, 2 : R, C \) and \( H \), are division algebras by the well known theorem of Frobenius.

Our aim is to survey different generalizations of the function theory of a complex variable which can be viewed as the study of those functions defined in a domain of \( R^2 \) and taking their values in the Clifford algebra \( R_{0,1} = C \).

The first key of the theory of holomorphic functions is, of course, the Cauchy-Riemann operator \( (\partial/\partial \bar{z} \text{ in the classical notations}) \), which can be written now as:

\[ D = \frac{\partial}{\partial x_0} + e_1 \frac{\partial}{\partial x_1}, \]

omitting the famous normalization constant 1/2. It should be noted that the definition domain of \( f \) lies in a space whose elements are couples of a scalar and a vector, so that \( R^2 \) should be identified to \( R \oplus iR \).

### 2. MONOGENIC FUNCTIONS

Let \( R_{0,n} \) be the Clifford algebra of the real vector space \( V \) of dimension \( n \) provided with a quadratic form of negative signature, \( n \in \mathbb{N} \). denote by \( S \) the set of the scalars in \( R_{0,n} \), identified with \( R \). Let \( \{e_i\}_{i=1}^n \) be an orthonormal basis of \( V \) and set also \( e_0 = 1 \).

A point \( x = (x_0, x_1, ..., x_n) \) of \( R^{n+1} \) will be considered as an element of \( S \oplus V \), namely \( x = \sum_{i=0}^{n} x_i e_i \). Such an element will be called a *paravector*. Obviously, it belongs to \( R_{0,n} \) and we can act on him with the conjugation: \( \bar{x} = x_0 - \sum_{i=0}^{n} x_i e_i \). It is remarkable that:

\[ x \bar{x} = \bar{x} x = |x|^2, \]
where \(|x|\) denotes the usual euclidean norm of \(x\) in \(\mathbb{R}^{n+1}\) and it shows that every non-zero paravector is invertible. Sometime, if necessary, we will resort to the notation \(x = x_0 + \overrightarrow{x}\), where \(\overrightarrow{x}\) is the vector part of \(x\), i.e. \(\overrightarrow{x} = \sum_{i=1}^{n} x_i e_i\).

Let \(f : \Omega \to \mathbb{R}_{0,n}\), where \(\Omega\) is an open subset of \(S \oplus V\). Introduce the Cauchy-Dirac-Fueter operator:

\[
D = \sum_{i=0}^{n} e_i \frac{\partial}{\partial x_i}
\]

Note \(D\) possesses a conjugate operator \(\overline{D} = \frac{\partial}{\partial x_0} - \sum_{i=1}^{n} e_i \frac{\partial}{\partial x_i}\) and that \(\overline{DD} = \overline{DD} = \Delta\), where \(\Delta\) is the usual Laplacian.

**Definition ([2])**. A function \(f : \Omega \to \mathbb{R}_{0,n}\), of class \(C^1\), is said to be (left) monogenic in \(\Omega\) if and only if \(Df(x) = 0\) for each \(x \in \Omega\).

Obviously, in the case \(n = 1\), we get the holomorphic functions of one complex variable.

**Important remark.** If \(n > 1\), then the functions \(x \mapsto x\) and \(x \mapsto x^m\), \(x \in S \oplus V, m \in \mathbb{N}\) are not monogenic.

Following R. Brackx, R. Delanghe and F. Sommen [2], recall that there exists a Cauchy kernel: \(E(x) = (\omega_n^{-1})(\overline{x}/|x|^{n+1})\) for \(x \in S \oplus V - \{0\}\), where \(\omega_n\) is the area of the unit sphere in \(\mathbb{R}^{n+1}\). This kernel is really well adapted to the monogenic functions because it is himself a monogenic function with a singularity at the origin, i.e. \(DE(x) = \delta\) for \(x \in S \oplus V\).

Then, put \(\omega(y) = dy_0 \wedge ... \wedge dy_n\) and \(\gamma(y) = \sum_{i=0}^{n} (-1)^i e_i dy_0 \wedge ... \wedge dy_i \wedge ... \wedge dy_n\). Thus, we have:

**Integral representation formula:** If \(f\) is monogenic in \(\Omega\) and \(U\) is an oriented compact differentiable variety of dimension \(n+1\) with boundary \(\partial U\) and \(U \subset \Omega\), then

\[
\int_{\partial U} E(y - x) \gamma(y) f(y) = f(x), \quad x \in \hat{U}.
\]

Thanks to this, the analogous of the mean value theorem, the maximum modulus principle, Morera’s theorem follow easily.

The depth and the wealth of the one complex variable theory come also thanks to the "duality": Cauchy-Riemann and Weierstrass, i.e. every holomorphic function is analytic and the reciprocate. How to understand what is the generalization of a power series?

In the frame of monogenic functions, an answer exists because, fortunately, the functions \(w_k = x_k e_0 - x_0 e_k, k = 1, ..., n\) are monogenic. Omitting the details and roughly speaking one can expand
every monogenic function in a series of polynomials which elementary monomials are the $w_k$ and their powers. Just for an illustration let us show this phenomena is somehow natural: suppose $f : S \oplus V \rightarrow \mathbb{R}_{0,n}$ is real analytic on a neighborhood of the origin, so

$$f(h) = \sum_{k=0}^{\infty} \left( h_0 \frac{\partial}{\partial x_0} + \ldots + h_n \frac{\partial}{\partial x_n} \right)^k f(0).$$

But at the same time $f$ is monogenic, i.e.

$$\frac{\partial f}{\partial x_0} = - \sum_{i=1}^{n} e_i \frac{\partial f}{\partial x_i}.$$

Hence

$$f(h) = \sum_{k=0}^{\infty} \left( \sum_{i=1}^{n} (h_i - e_i h_0) \frac{\partial}{\partial x_i} \right)^k f(0).$$

3. HOLOMORPHIC CLIFFORDIAN FUNCTIONS

Here, consider functions $f : \Omega \rightarrow \mathbb{R}_{0,2m+1}$, where $\Omega$ is an open subset of $S \oplus V = \mathbb{R} \oplus \mathbb{R}^{2m+1} = \mathbb{R}^{2m+2}$. The paravectors of $S \oplus V$ will be written as $x = x_0 + \vec{x}$, $x_0 \in \mathbb{R}$, $\vec{x} = \sum_{i=1}^{2m+1} x_i e_i$.

**Definition** ([11], [12]). A function $f : \Omega \rightarrow \mathbb{R}_{0,2m+1}$ is said to be (left) *holomorphic Cliffordian* in $\Omega$ if and only if:

$$D \Delta^m f(x) = 0$$

for each $x \in \Omega$. Here $\Delta^m$ means the iterated Laplacian.

The set of holomorphic Cliffordian functions is wider than those of the monogenic ones: *every monogenic is also holomorphic Cliffordian, but the reciprocal is false*. Indeed, if $Df = 0$, then $D \Delta^m f = \Delta^m Df = 0$ because $\Delta^m$ is a scalar operator. The simplest example of a holomorphic Cliffordian function which is not monogenic is the identity $x \mapsto x$. Actually, one can prove that all entire powers of $x$ are holomorphic Cliffordian, while they are not monogenic.

Note that $f$ is holomorphic Cliffordian if and only if $\Delta^m f$ is monogenic.

There is a simple way to construct holomorphic Cliffordian functions which is based on the Fueter principle [7].

**Lemma 1.** If $u : \mathbb{R}^{2m+2} \rightarrow \mathbb{R}$ is harmonic, then $u(x_0, | \vec{x} |)$, where $x = x_0 + \vec{x}$ and $| \vec{x} |^2 = \sum_{i=1}^{2m+1} x_i^2$ is $(m+1)$-harmonic, i.e.

$$\Delta^{m+1} u(x_0, | \vec{x} |) = 0.$$
Lemma 2. If \( f : (\xi, \eta) \mapsto f = u + iv \) is a holomorphic function, then \( F(x) = u(x_0, \mid x \mid) + \frac{\overrightarrow{x}}{\mid x \mid} v(x_0, \mid x \mid) \) is a holomorphic Cliffordian function.

If we summarize: from an usual holomorphic function \( f \), with real part \( u \), we construct the associated \((m+1)\)-harmonic \( u(x_0, \mid \overrightarrow{x} \mid) \) and then it suffices to take \( Du(x_0, \mid \overrightarrow{x} \mid) \) in order to get a holomorphic Cliffordian one. This receipt is very well adapted for the construction of trigonometric or exponential functions in \( \mathbb{R}_{0,2m+1} \), [15]. Note also the set of holomorphic Cliffordian functions is stable under any directional derivation.

It is natural to ask for an integral representation formula but in this case, the operator \( D\Delta^m \) being of order \( 2m + 1 \), such a formula would be much more complicated. Anyway, the first step is to exhibit an analogous to the Cauchy kernel. Remember the fundamental solution of the iterated Laplacian \( \Delta^{m+1} h(x) = 0 \) for \( x \in S \oplus V - \{0\} \) is known: that is \( h(x) = ln \mid x \mid \). Hence \( \overline{D} \left( \frac{1}{2}ln(x\overline{x}) \right) \) must be holomorphic Cliffordian. But:

\[
\overline{D} \left( \frac{1}{2}ln(x\overline{x}) \right) = \frac{1}{2} \frac{\overline{D}(\mid x \mid^2)}{\mid x \mid^2} = \frac{\overline{x}}{\mid x \mid^2} = x^{-1}.
\]

By the way, we found the first holomorphic Cliffordian function with an isolated punctual singularity at the origin.

It is remarkable that, after computations, one get: \( \Delta^m(x^{-1}) = (-1)^m 2^m (m!)^2 \omega_m E(x) \). It becomes natural to introduce a new kernel:

\[
N(x) = \varepsilon_m x^{-1},
\]

with the suitable choice for the constant \( \varepsilon_m = (-1)^m [2^{2m+1}m!(m+1)]^{-1} \).

So, \( N \) is the natural Cauchy kernel for holomorphic Cliffordian functions and we have:

\[
D\Delta^m N(x) = D E(x) = \delta, \quad x \in S \oplus V.
\]

**Integral representation formula.** Let \( B \) be the unit ball in \( \mathbb{R}_{2m+2} \), \( x \) an interior point of \( B \), \( \frac{\partial}{\partial n} \) means the derivation in the direction of the outward normal.Thus, we have:

\[
f(x) = \int_{\partial B} (\Delta^m N(y - x)\gamma(y)f(y)
- \sum_{k=1}^{m} \int_{\partial B} \left( \frac{\partial}{\partial n} \Delta^{m-k} N(y - x) \right) D\Delta^{k-1} f(y) d\sigma_y
+ \sum_{k=1}^{m} \int_{\partial B} (\Delta^{m-k} N(y - x)) \frac{\partial}{\partial n} D\Delta^{k-1} f(y) d\sigma_y.
\]
The above formula involves $2m + 1$ integrals on $\partial B$, which means one can deduce the values of $f$ inside $B$ knowing the values on $\partial B$ of $f, D\Delta^{k-1}f$ and $\frac{\partial}{\partial n} D\Delta^{k-1}f$, \( k = 1, 2, ..., m \).

Recall that all integer powers of a paravector $x$ are solutions of $D\Delta^m = 0$, and we saw also that $D\Delta^m(x^{-1}) = \delta$. Those facts can be proved directly following straightforward computations. Let $x = x_0 + \overline{x}$ be a paravector in a general Clifford algebra of antieuclidean type $\mathbf{R}_{0,d}, d \in \mathbb{N}$. Very fastidious calculations give:

$$D\Delta^m(x^{-1}) = (-1)^m 2^m m! \prod_{j=0}^{m} (2j + 1 - d)(|x|^{-2m+2}).$$

The right hand side is a scalar vanishing for $d = 1, 3, 5, ..., 2m + 1$. Moreover, we have:

$$D\Delta^m(x^{2n+1}) = \prod_{j=0}^{m} (2j + 1 - d) \prod_{q=m}^{n} \prod_{k=1}^{m} (2q + 2k - 2) e^{2q+1} x_0^{2n-2q} \overline{x}^{2q-2m},$$

and a similar formula for an even power $x^{2n}$ of $x$. In both cases, the right hand sides are again scalars vanishing for $d = 1, 3, 5, ..., 2m + 1$. 

**Polynomial solutions of $D\Delta^m = 0$.** Set $\alpha = (\alpha_0, \alpha_1, ..., \alpha_{2m+1}), \alpha_j \in \mathbb{N}, |\alpha| = \sum_{j=0}^{2m+1} \alpha_j$. Consider the set

$$\{e_\nu\} = \{e_0, ..., e_0, e_1, ..., e_1, ..., e_{2m+1}, ..., e_{2m+1}\},$$

where $e_0$ is written $\alpha_0$ times, $e_i$ : $\alpha_i$ times. Then set:

$$P_\alpha(x) = \sum_{\Theta} \prod_{\nu=1}^{|\alpha|-1} (e_{\sigma(\nu)} x) e_{\sigma(|\alpha|)},$$

the sum being expanded over all distinguishable elements $\sigma$ of the permutation group $\Theta$ of the set $\{e_\nu\}$. The $P_\alpha$ are polynomials of degree $|\alpha| - 1$. A straightforward calculation carried on them shows that $P_\alpha$ is equal up to a rational constant to $\partial^{||\alpha||} |x^{2||\alpha||-1}|$. Thus, it follows that the $P_\alpha$ are holomorphic Cliffordian functions, which are left and right, thanks to the symmetrization process.

The classical way for getting Taylor’s series of a holomorphic function is to expand the Cauchy kernel in the integral representation formula. The same procedure is available here:

$$(y - x)^{-1} = (y(1 - y^{-1}x))^{-1} = (1 - y^{-1}x)y^{-1} =$$

$$= y^{-1} + y^{-1}xy^{-1} + y^{-1}xy^{-1}xy^{-1} + ... + (y^{-1}x)^n y^{-1} + ...$$

Obviously, we have $y^{-1} = \bar{y}(|y|)^{-2}$, and thus:

$$(y - x)^{-1} = \sum_{k=0}^{\infty} \frac{(\bar{y}x)^k \bar{y}}{|y|^{2k+2}}.$$

7
It is not difficult to observe the polynomials $P_\alpha$ appear again. Finally, as in the classical case, we can deduce the expansion in a "power series" of any holomorphic Cliffordian function $f$ under the form:

$$f(x) = \sum_{|\alpha|=1}^{\infty} P_\alpha(x)C_\alpha,$$

where $C_\alpha \in \mathbb{R}_{0,2m+1}$.

Further, let us mention that a function $f$ which is holomorphic Cliffordian in a punctured neighborhood of the origin possesses a Laurent expansion ([11], [12]). So, the set of meromorphic Cliffordian functions with isolated singularities is well defined.

What about the case $\mathbb{R}_{0,2m}$? More precisely, could we pass from holomorphic Cliffordian functions in $\mathbb{R}_{0,2m+1}$ to their restrictions in $\mathbb{R}_{0,2m}$? Roughly speaking, it is the same as between $\mathbb{C}$ and $\mathbb{R}$. In the general case of a Clifford algebra of type $\mathbb{R}_{0,d}$, we can observe that the set of homogeneous polynomials of degree $n$ which are holomorphic Cliffordian is a right $\mathbb{R}_{0,d}$-module generated by the monomials $(ax)^n a$, where $a$ is a paravector.

Now, let us consider $f : \mathbb{R} \oplus \mathbb{R}^d \to \mathbb{R}_{0,d}$.

Case 1: $d$ is odd. The function $f$ will be holomorphic Cliffordian if $D\Delta^{d+1} f = 0$.

Case 2: $d$ is even. We say that $f$ is analytic Cliffordian ([10]) if there is a holomorphic Cliffordian function of one more variable $F : \mathbb{R} \oplus \mathbb{R}^{d+1} \to \mathbb{R}_{0,d+1}$, which is even with respect to $e_{d+1}$ and such that $F |_{x_{d+1}=0} = f$.

4. HYPERMONOGENIC FUNCTIONS

In this part we will briefly discuss another class of functions, named hypermonogenic, which were introduced by H. Leutwiler ([16], [17]) and studied by himself and Sirkka-Liisa Eriksson-Bique ([6]).

Recall that any element $a \in \mathbb{R}_{0,n}$ may be uniquely decomposed as $a = b + ce_n$ for $b, c \in \mathbb{R}_{0,n-1}$. This should be compared with the classical decomposition of a complex number $a = b + ic$. Using the above decomposition, one introduces the projections $P : \mathbb{R}_{0,n} \to \mathbb{R}_{0,n-1}$ and $Q : \mathbb{R}_{0,n} \to \mathbb{R}_{0,n-1}$ given by $Pa = b, Qa = c$.

Now define the following modification of the Dirac operator $D$ as follows:

$$Mf = Df + \frac{n-1}{x_n} (Qf)^*,$$

where $^*$ denotes the automorphism introduced above.
Definition. An infinitely differentiable function \( f : \Omega \rightarrow \mathbb{R}_{0,n}, \) \( \Omega \) being an open subset of \( \mathbb{R}^{n+1} \), such that \( Mf = 0 \) on \( \Omega - \{ x : x_n = 0 \} \) is called a (left) hypermonogenic function.

Decomposing \( f \) into \( f = Pf + (Qf)e_n \), its \( P \)-part satisfies the Laplace-Beltrami equation:

\[
x_n \Delta(Pf) - (n - 1) \frac{\partial(Pf)}{\partial x_n} = 0,
\]

associated to the hyperbolic metric, defined on the upper half space \( \mathbb{R}^{n+1}_+ \) by \( ds^2 = x_n^{-2}(dx_0^2 + dx_1^2 + \ldots + dx_n^2) \).

Its \( Q \)-part solves the eigenvalue equation:

\[
x_n^2 \Delta(Qf) - (n - 1)x_n \frac{\partial(Qf)}{\partial x_n} + (n - 1)Qf = 0.
\]

It turns out that hypermonogenic functions are stable by derivations in all possible directions excepted this one on \( x_n \) and that, for any \( m \in \mathbb{N} \), the maps \( x \mapsto x^m \) and \( x \mapsto x^{-m} \) are hypermonogenic in \( \mathbb{R}^{n+1} \), resp. \( \mathbb{R}^{n+1}_+ \).

Clearly, hypermonogenic functions generalize usual holomorphic functions of a complex variable. But what about the relations of this class with the class of holomorphic Cliffordian?

One can prove that every hypermonogenic function is also a holomorphic Cliffordian function. Let us study this problem in the case \( n = 3 \).

Assuming \( f : \Omega \rightarrow \mathbb{R}_{0,3} \) is hypermonogenic, \( Df = -\frac{2}{x_3} (Qf)^* \) and hence \( D(\Delta f) = -2 \Delta \left[ \frac{(Qf)^*}{x_3} \right] \). An explicit calculation of the last expression combined with the above eigenvalue equation for the \( Q \)-part of \( f \), allow us to conclude that

\[
\Delta \left[ \frac{(Qf)^*}{x_3} \right] = \frac{1}{x_3} [\Delta(Qf) - \frac{2}{x_3} \frac{\partial(Qf)}{\partial x_3} + 2 \frac{Qf}{x_3^2}]^* = 0
\]

forcing \( D(\Delta f) = 0 \).

Let say also the theory of hypermonogenic functions is provided with an integral representation formula and that the expansion in power series is generated by polynomials which are deeply related to the \( P_\alpha \) above.

However, it should be noted an important difference: multiplication of \( e_n \) from the right to a hypermonogenic function does not in general give again a hypermonogenic function. In the larger class of holomorphic Cliffordian functions, this operation is allowed, even multiplication from the right with any Clifford number.
5. ELLIPTIC CLIFFORDIAN FUNCTIONS

One of the most stimulating aspects of the theory of functions of a complex variable is the theory of elliptic functions. In [13], [14], we tried to put the foundations of their Cliffordian analogues. However, the main difficulty to overcome here is the lack of a satisfactory notion of product of two holomorphic Cliffordian functions. Thus, we had to adapt only the “additive” part of the theory in our case. Fortunately, the way drawn by Weierstrass, using the $\zeta$ and then the $\mathcal{P}$ function, was the right one. It was amazing that the analogues of the Jacobi functions $cn$, $dn$ and $sn$ could also be find on the same way.

Let us make a remark: If $f$ is real-analytic in a neighborhood $W$ of $a \in S \oplus V$ and is taking its values in $S \oplus V$, then $f$ admits a Taylor expansion:

$$f(a + h) = \sum_{n=0}^{\infty} \frac{1}{n!}(h \mid \nabla_x)^n f(x) \mid_{x=a},$$

where $(h \mid \nabla_x)$ is the scalar product in $\mathbb{R}^{2m+2}$. Note also that $(h \mid \nabla_x)(x) = h$, $(h \mid \nabla_x)(x^{-1}) = -x^{-1}hx^{-1}$, and that

$$(h \mid \nabla_x)^q(x^{-1}) = (-1)^q q!(x^{-1}h)^q x^{-1}, \ q \in \mathbb{N}.$$

Now, let $N \in \{1, 2, ..., 2m+2\}$ and $\omega_\alpha \in S \oplus V$ be paravectors when $\alpha = 1, 2, ..., N$. Suppose the $\omega_\alpha$ to be $\mathbb{R}$-independants. A function $f : \Omega \to \mathbb{R}$ is said to be $N$-periodic if $f(x + 2\omega_\alpha) = f(x)$ for $x \in S \oplus V$ and $\alpha = 1, 2, ..., N$. The associated lattice is: $2\mathbb{Z}^N \omega = \{2k\omega, k \in \mathbb{Z}^N\}$, where $\omega = (\omega_1, ..., \omega_N)$ is the generic notation for a half period and $k\omega = \sum_{\alpha=1}^{N} k_\alpha \omega_\alpha$. Rewrite the lattice in $\{w_p\}_{p=0}^{\infty}$, where $w_0 = (0, ..., 0)$.

**Definition.** Introduce the $\zeta_N$ Weierstrass functions as:

$$\zeta_N(x) = x^{-1} + \sum_{p=1}^{\infty} \{(x - w_p)^{-1} + \sum_{\mu=0}^{N-1} (w_p^{-1} x)^\mu w_p^{-1}\}.$$

Thus $\zeta_N : S \oplus V \setminus 2\mathbb{Z}^N \omega \to S \oplus V$ is a holomorphic Cliffordian function and possesses simple poles on the vertices of the lattice.

The function $\zeta_N$, is odd, is not itself a $N$-periodic function, but satisfies a property of ”quasi-periodicity”:

$$\zeta_N(x + 2\omega) - \zeta_N(x) = 2 \sum_{p=0}^{\lfloor \frac{N+1}{2} \rfloor-1} \frac{(x + \omega \mid \nabla_y)^{2p}}{(2p)!} \zeta_N(y) \mid_{y=\omega}.$$
which is equivalent to:

\[
\zeta_N(x + \omega) - \zeta_N(x - \omega) = 2 \sum_{p=0}^{[\frac{N+1}{2}]-1} \frac{(x | \nabla_y)^{2p}}{(2p)!} \zeta_N(y) \big|_{y=\omega}
\]

In particular, for \( N = 2m + 2 \), one has:

\[
\zeta_{2m+2}(x + \omega) - \zeta_{2m+2}(x - \omega) = 2 \sum_{p=0}^{m} \frac{(x | \nabla_y)^{2p}}{(2p)!} \zeta_{2m+2}(y) \big|_{y=\omega}
\]

which is the natural generalization of the well-known:

\[
\zeta(z + \omega) - \zeta(z - \omega) = 2\zeta(\omega).
\]

It is important to note that the right hand side of the previous equality is a holomorphic Cliffordian polynomial of degree \( 2m \).

As far as the Laurent expansion of \( \zeta_N \) in a neighborhood of the origin, it is easy to get it:

\[
\zeta_N(x) = x^{-1} + \sum_{k \geq \lceil \frac{N+1}{2} \rceil} \frac{1}{(2k + 1)!} \sum_{p=1}^{k} (x | \nabla_w)^{2k+1} (w^{-1}) \big|_{w=w_p}.
\]

Note that in the special case \( N = 2m + 2 \), the first sum starts from \( k = m + 1 \).

Now, in order to mimic the procedure of getting an elliptic function as \( P \) from \( \zeta \), where, in the case \( m = 0 \), one need only one derivation, here, differentiating \( 2m + 1 \) times \( \zeta_{2m+2} \), we are able to produce analogous of the \( P \) function. It is remarkable, that now, we can differentiate in all the directions of \( \mathbb{R}^{2m+2} \). The set of elliptic Cliffordian functions we are producing is quite larger than in the complex case.

Return now to the problem of the construction of analogous of the Jacobi functions. We have two problems to solve. Reduce the periods by half and eliminate the holomorphic Cliffordian polynomials appearing in the ”quasi-periodicity” property.

Introduce the so-called translation operators \( E_j \): for an arbitrary function \( \varphi : S \oplus V \to \mathbb{R}_{0,2m+1} \), set:

\[
E_j(\varphi)(x) = \varphi(x + \omega_j), \quad j = 1, 2, \ldots, N.
\]
When $\varphi$ is $N$-periodic, we can write:

$$(I - E_j^2)(\varphi)(x) = 0.$$  

As far as the "quasi-periodicity" of $\zeta_{2m + 2}$ is concerned, we can write it in the following form:

$$(I - E_j^2)(\zeta_{2m + 2})(x) = -p_{2m}(x; \omega_j),$$

for $j = 1, 2, ..., 2m + 2$, where $p_{2m}$ is the polynomial appearing in the formula of "quasi-periodicity".

How eliminate such a polynomial? In numerical analysis there is a nice recipe which says: when one want annihilate a polynomial of degree $d$, it suffices to apply $d + 1$ times operators of the form $I - E_j$ without being restricted to use always the same $j$.

Thus, for example, the following formula:

$$\prod_{j=1}^{2m + 1} (I - E_j)(I - E_j^2)(\zeta_{2m + 2})(x) = 0$$

is readable from the right to the left and says that $\zeta_{2m + 2}$ is quasi-periodic in the direction $2\omega_i$ and that after, we have proceeded to the elimination of the quasi-periodicity polynomial.

But the translation operators are commuting, so, the same formula can be written as:

$$(I - E_j^2)(\prod_{j=1}^{2m + 1} (I - E_j)(\zeta_{2m + 2})(x)) = 0$$

and it says that the function $\prod_{j=1}^{2m + 1} (I - E_j)(\zeta_{2m + 2})(x)$ is periodic with period $2\omega_i$.

Omitting some details, let set now:

$$C(x) = \prod_{j=1}^{2m + 2} (I - E_j)(\zeta_{2m + 2})(x)$$

$$S_i(x) = (I + E_i) \prod_{j=1, j \neq i}^{2m + 2} (I - E_j)(\zeta_{2m + 2})(x),$$
where \( i = 1, 2, \ldots, 2m + 2 \).

Here, we have made use of the operator \( I + E_i \) which reduces the periodicity from \( 2\omega_i \) by half to \( \omega_i \).

So, we have \( 2m + 3 \) elliptic Cliffordian functions whose periods are \( \{ \omega_i + \omega_k \}, i, k = 1, 2, \ldots, 2m + 2 \) for \( C(x) \), and

\[
2\omega_1, \ldots, 2\omega_{i-1}, \omega_i, 2\omega_{i+1}, \ldots, 2\omega_{2m+2}
\]

for \( S_i(x) \). The way they were obtained obeys to strong lays, they are no more, no less than \( 2m + 3 \) and they are obviously the natural generalizations in \( \mathbb{R}^{2m+2} \) of the classical Jacobi functions \( cn, dn, sn \) (in this order!) we can get when \( m = 0 \).

6. GENERALIZED AUTOMORPHIC FORMS

The story started with the problem: how to generalize the notion of modular forms in the case of functions with values in a Clifford algebra?

Historically, an enormous work has been done in some previous papers ([8], [3]), first in the case of monogenic functions and further, in the case of \( k \)-hypermonogenic functions. (The definition of such a function will be given below).

Although a technical virtuosity displayed in this situations, the results were finally unsatisfying in some sense: actually, the authors have not been able to propose a construction for non-vanishing \( k \)-hypermonogenic cusp forms for \( k \neq 0 \).

Recently, in [4], the authors decided to consider a larger class of functions that contains the class of \( k \)-hypermonogenic functions as a special subset.

When \( k \in \mathbb{Z} \) is even, the class of considered functions, named \( k \)-holomorphic Cliffordian, are those which are annihilated by the operator \( D\Delta^{\frac{k}{2}} \), where \( D \) is the Dirac operator, \( \Delta \) is the Laplacian and \( \Delta^{\frac{k}{2}} \) the iterated Laplacian. For \( k = n - 1 \) and \( n \) odd this is the class of the holomorphic Cliffordian functions.

Finally, it turned out that the choice of the authors to consider \( k \)-holomorphic Cliffordian functions was fruitful. The main obstruction they met before, namely, multiplication of \( e_n \) from the right to a \( k \)-hypermonogenic function which does not in general give again a \( k \)-hypermonogenic function, was overcome in the new context: right multiplication of any Clifford number with a \( k \)-holomorphic Cliffordian function remains \( k \)-holomorphic Cliffordian. It is amazing that now all the machinery works perfectly, even elegantly.
The general Ahlfors-Vahlen group and some discrete arithmetic subgroups.

The set that consists of Clifford valued matrices \[
\begin{pmatrix}
a & b \\
c & d \\
\end{pmatrix}
\]
whose coefficients satisfy the conditions below forms a group under matrix multiplication. It is called the general Ahlfors-Vahlen group \(G_{AV}(\mathbb{R} \oplus \mathbb{R}^n)\). The action of this group on \(\mathbb{R} \oplus \mathbb{R}^n\) is described by the associated Möbius transformation. We can also restrict this action on the upper half-space \(H^+(\mathbb{R} \oplus \mathbb{R}^n) = \{x \in \mathbb{R} \oplus \mathbb{R}^n : x_n > 0\}\) in the following way:

\[
G_{AV}(\mathbb{R} \oplus \mathbb{R}^n) \times H^+(\mathbb{R} \oplus \mathbb{R}^n) \rightarrow H^+(\mathbb{R} \oplus \mathbb{R}^n)
\]

by \((M, x) \mapsto M < x > = (ax + b)(cx + d)^{-1}\). Here, the coefficients \(a, b, c, d\) from \(\mathbb{R}_{0,n}\) satisfy:

(i) \(a, b, c, d\) are products of paravectors
(ii) \(\bar{a}d - \bar{b}c \in \mathbb{R} \setminus \{0\}\)
(iii) \(ac^{-1}, c^{-1}d \in \mathbb{R}^{n+1}\) for \(c \neq 0\) and \(bd^{-1} \in \mathbb{R}^{n+1}\) for \(c = 0\).

The subgroup consisting of those matrices from \(G_{AV}(\mathbb{R} \oplus \mathbb{R}^n)\) that satisfy \(ad^* - bc^* = 1\) is called the special Ahlfors-Vahlen group, denoted by \(SAV(\mathbb{R} \oplus \mathbb{R}^n)\).

The automorphism group of the upper half-space \(H^+(\mathbb{R} \oplus \mathbb{R}^n)\) is the group \(SAV(\mathbb{R} \oplus \mathbb{R}^{n-1})\).

The rational Ahlfors-Vahlen group \(SAV(\mathbb{R} \oplus \mathbb{R}^{n-1}, \mathbb{Q})\) is the set of matrices \(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\) from \(SAV(\mathbb{R} \oplus \mathbb{R}^{n-1})\) that satisfy

(i) \(a\bar{a}, b\bar{b}, c\bar{c}, d\bar{d} \in \mathbb{Q}\),
(ii) \(a\bar{c}, b\bar{d} \in \mathbb{Q} \oplus \mathbb{Q}^n\),
(iii) \(a\bar{x} + b\bar{x}\bar{a}, c\bar{x}\bar{d} + d\bar{x}\bar{c} \in \mathbb{Q}\),
(iv) \(a\bar{x} + b\bar{x}\bar{c} \in \mathbb{Q} \oplus \mathbb{Q}^n\)

The following definition provides us with a whole class of arithmetic subgroups of the Ahlfors-Vahlen group which act totally discontinuously on the upper half-space.

**Definition:**

\[
\Gamma_{n-1}(\mathcal{I}) = SAV(\mathbb{R} \oplus \mathbb{R}^{n-1}, \mathbb{Q}) \cap Mat(2, \mathcal{I}).
\]

For \(N \in \mathbb{N}\), set:

\[
\Gamma_{n-1}(\mathcal{I})[N] = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_{n-1}(\mathcal{I}), a - 1, b, c, d - 1 \in N\mathcal{I} \right\}
\]

14
Here \( \mathcal{I} \) is a \( \mathbb{Z} \)-order in the Clifford algebra which is roughly speaking a subring \( R \) such that the additive group of \( R \) is finitely generated and contains a \( \mathbb{Q} \)-basis of the algebra.

Notice that all the groups \( \Gamma_{n-1}(\mathcal{I})[N] \) are discrete groups and act totally discontinuously on the upper half-space.

Concerning the above notions, see [1], [3], [4], [8] for the details.

**Monogenic functions.** An important property of the Dirac operator

\[
D = \frac{\partial}{\partial x_0} + \frac{\partial}{\partial x_1} e_1 + \ldots + \frac{\partial}{\partial x_n} e_n
\]

is its quasi-invariance under Möbius transformations acting on the complete Euclidean space \( \mathbb{R} \oplus \mathbb{R}^n \).

Let \( M \in GAV(\mathbb{R} \oplus \mathbb{R}^n) \) and \( f \) be a left monogenic function in the variable \( y = M < x > = (ax + b)(cx + d)^{-1} \). Then the function:

\[
g(x) = \frac{cx + d}{||cx + d||^{n+1}} f(M < x >)
\]

is again left monogenic on \( x \) for any \( M \in GAV(\mathbb{R} \oplus \mathbb{R}^n) \)

**k-hypermonogenic functions.** The class of monogenic functions belongs to the more general class of so-called \( k \)-hypermonogenic functions. They are defined as the null-solutions to the system:

\[
Df + k \frac{(Qf)^*}{x_n} = 0,
\]

where \( k \in \mathbb{R} \).

Note that in the case of \( k = 0 \), we are dealing with left monogenic functions. The particular case of \( k = n - 1 \) corresponds to hypermonogenic functions.

Now, if \( f \) is \( k \)-hypermonogenic in the variable \( y = M < x > = (ax + b)(cx + d)^{-1} \), then:

\[
F(x) = \frac{cx + d}{||cx + d||^{n+1-k}} f(M < x >)
\]

is \( k \)-hypermonogenic.

However, this invariance holds only for matrices from
This is due to the fact that a translation in the argument of a $k$-hypermonogenic function in the $e_n$-direction does not give a $k$-hypermonogenic function again.

As we said before, the authors [4] decided to consider a larger class of functions, containing in itself the set of $k$-hypermonogenic ones and possessing the previous property (being stable under right multiplication with $e_n$). Thus, this new class will have the extra property of being invariant under the whole group $\mathcal{S}AV(\mathbb{R} \oplus \mathbb{R}^n)$.

**$k$-holomorphic Cliffordian functions.**

**Definition.** Let $n \in \mathbb{N}$ and suppose that $k$ is an even positive integer. Let $U \subset \mathbb{R} \oplus \mathbb{R}^n$ be an open set. Then we call a function $f : U \to \mathbb{R}_{0,n}$ a $k$-holomorphic Cliffordian function if

$$D \Delta^{\frac{k}{2}} f = 0.$$  

In the particular case of $k = n - 1$ ($n$ odd) we deal with the class of holomorphic Cliffordian functions.

One can also introduce $k$-holomorphic Cliffordian functions for negative even integers. This can be done through the Teodorescu transform which is the right inverse to $D$, i.e. $DTf = 0$. In view of the identity $\bar{D}D = \Delta$, one can formally express $\Delta^{-1}$ as $\bar{T}T$ on the upper half-space.

First of all, it is easy to prove that, for any even $k \in \mathbb{Z}$, every $k$-hypermonogenic function is also $k$-holomorphic Cliffordian.

As a consequence, the $k$-hypermonogenic kernel functions

$$G_k(x) = \frac{x}{\|x\|^{n+1-k}}$$

are also $k$-holomorphic Cliffordian for all $k \in 2\mathbb{Z}$.

However, note that not every $k$-holomorphic Cliffordian function is $k$-hypermonogenic. Take for instance $k = n - 1$, with $n$ odd. Then the functions $xe_n$ and $x + e_n$, where $x$ is a paravector, are both holomorphic Cliffordian but not hypermonogenic.

**Theorem.** Let $k \in 2\mathbb{Z}$. Suppose that $M \in SAV(\mathbb{R} \oplus \mathbb{R}^n)$. Let $y = M < x > = (ax + b)(cx + d)^{-1}$ be the image of a point $x$
under such a Möbius transformation. Then such a function $f(y)$ that is $k$-holomorphic Cliffordian in the variable $y$ is transformed to a function

$$F(x) = \frac{cx + d}{||cx + d||^{n+1-k}}f(M < x >)$$

which turns out to be $k$-holomorphic Cliffordian in the variable $x$.

**Definition.** Let $p \leq n - 1$ and suppose that $k \in 2\mathbb{Z}$. A left $k$-holomorphic Cliffordian function $f : H^+(\mathbb{R} \oplus \mathbb{R}^n) \rightarrow \mathbb{R}_{0,n}$ is called a **left $k$-holomorphic Cliffordian automorphic form on** $\Gamma_p(I)[N]$, if for all $x \in H^+(\mathbb{R} \oplus \mathbb{R}^n)$

$$f(x) = \frac{cx + d}{||cx + d||^{n+1-k}}f(M < x >)$$

for all $M \in \Gamma_p(I)[N]$.

In the case $k = 0$ we re-obtain the class of left monogenic automorphic forms discussed in [8].

Moreover, all $k$-hypermonogenic automorphic forms discussed in [3] are included in this set.

The simplest examples of $k$-holomorphic Cliffordian automorphic forms on the groups $\Gamma_p(I)[N]$ are the generalized Eisenstein series given in [8], as well as the simplest examples of $k$-holomorphic Cliffordian automorphic forms for discrete translation groups for the special case of $k = n - 1$ with $n$ odd, were given in [13] (the generalizations of the cotangent function, the Weierstrass $\zeta$ and $\wp$ functions).

**Cusp forms**

**Definition:** For even integers $k \leq 0$ a left $k$-holomorphic Cliffordian cusp form on $\Gamma_{n-1}(I)[N]$ is a left $k$-holomorphic Cliffordian automorphic form on $\Gamma_{n-1}(I)[N]$ that satisfies additionally:

$$\lim_{x_n \rightarrow +\infty} x_n^{-k} \frac{cx_n e_n + d}{||cx_n e_n + d||^{n+1-k}}f(M < x_n e_n >) = 0$$

for all $M \in \Gamma_{n-1}(I)[N]$.

For positive even integers $k$, the factor $x_n^{-k}$ is omitted.

In the last part of the paper [4], the authors establish a surprising result: a decomposition theorem of the spaces of $k$-holomorphic Cliffordian automorphic forms in terms of a direct orthogonal sum of the spaces of $k$-hypermonogenic Eisenstein series and of $k$-holomorphic Cliffordian cusp forms.
Let us end with a citation due to L. Ahlfors in [1]: "The aim is not to prove new results, but to try to convince complex analysts of more traditional bent that the use of Clifford numbers is both natural, simple and useful."

References


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